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The Hilbert space $L^2(SU(2))$ as a representation space for the group $SU(2) \times SU(2)$

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Abstract. The Hilbert space $L^2(SU(2))$ is used as a representation space for a (unitary) representation of the direct product group $SU(2) \times SU(2)$ and the corresponding group algebra. Three different types of operators which are closely related to the representation theory of SU(2) are used to construct convenient operator bases whose elements are irreducible tensor operators with respect to $SU(2) \times SU(2)$. A complete set of irreducible tensor operators and useful operator identities are derived.

1. Introduction

A topic of considerable interest in mathematical physics is carrier spaces which are symmetric homogeneous spaces over some groups G. It is well known that such Hilbert spaces (invariant subspaces or coset spaces) turn up in various physical problems. Malin (1975) discussed the Weyl and Dirac equations in terms of functions over the group SU(2); Šijački (1975) used such a Hilbert space as a representation space for the group SL(3, R), whose representation theory was successfully applied to nuclear rotational motion (Weaver and Biedenharn 1972) and to strong quantum gravitational fields in general relativity (Rosen 1966); Beers and Millman (1975) discussed convergence problems connected with the multipole expansion of radiation fields using SU(2) as the domain of the vector (tensor) harmonics; Hu (1973, 1974) showed that the solutions of the Helmholtz equation for the Mixmaster universe in general relativity are equivalent to that of the quantum mechanical problem of the asymmetric rotator, whose carrier space is just the coset space $L^2(SO(3))$; and Marshalek (1975) discussed the asymmetric-top model in nuclear physics, where a special class of irreducible tensor operators, which will be considered later, is used to represent the transition operator in terms of creation and destruction operators. Therefore, with regard to the physical applications, part of the problem consists in defining appropriately a unitary representation of a group G' in a given Hilbert space, here $L^2(G)$, and another part in solving the problem of how to label functions and/or operators with respect to G'. Thereby we understand by the labelling problem the construction of functions and/or operators which transform according to the unitary irreducible representations (unirreps) of G'. The choice of the group G' depends on the applications which one wants to consider. Besides this, for the labelling problem all subduction matrices belonging to a given chain of subgroups (depending on the physical problem) should also be well known.

In the present paper we choose G as SU(2) and discuss the labelling problem for the direct product group $G' = \mathrm{SU}(2) \times \mathrm{SU}(2)$. What we are primarily interested in is to construct convenient operator bases for the linear operator space containing all operators defined in $L^2(\mathrm{SU}(2))$. The main purpose of this paper is to develop a mathematical framework which allows us to express any interaction operator (defined in $L^2(\mathrm{SU}(2))$, or in an invariant subspace or in a coset space of the former) in terms of a symmetry-adapted operator basis. Problems of this kind are quite common in physical applications, like the vector harmonic expansions of radiation fields discussed by Beers and Millman (1975). Symmetry-adapted operator bases are bases whose elements are irreducible tensor operators (IT) with respect to the group in question. Furthermore, we require that the matrix elements of these operators are easily calculated and that this basis consists of operators which arise in many physical problems. Operators of this type are the elements of the group algebra, of the Lie or enveloping algebra, or operators defined by matrix elements of unirreps (such as the spherical harmonic operators).

The material is organized as follows. In § 2 we define a unitary representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and a representation of the corresponding group algebra. In § 3 we give the general definition of an irreducible tensor operator with respect to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In § 4 we introduce IT within the group algebra and discuss in § 5 IT which are closely related to the matrix elements of the unirreps of SU(2). Operator identities between special IT are derived in § 6 where combinations of IT of both the preceding sections are considered. We can show that three different types of these IT are essential. For one of these types it can be shown that the operators form a complete set of IT so that every operator $\mathcal{O}$ in $L^2(\mathrm{SU}(2))$ can be expressed as a unique linear combination of these IT. In § 8 we discuss a special class of IT consisting of IT defined in § 5 and elements of the enveloping algebras of SU(2). Such operators nearly always occur in physical applications.

2. $L^2(\mathrm{SU}(2))$ as a representation space for $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the corresponding group algebra

The separable Hilbert space $L^2(\mathrm{SU}(2))$ is the set of all complex-valued square integrable functions where the scalar product is given by

$$\langle f, h\rangle = \int \mathrm{d}\mu(\omega) f^*(\omega) h(\omega) \tag{2.1}$$

$$\mu(\mathrm{SU}(2)) = 1. \tag{2.2}$$

The set of elements

$$\{\mathcal{D}^j_{mk} = (2j+1)^{1/2} D^{j*}_{mk}\colon j = 0, \tfrac{1}{2}, 1, \ldots; -j \leqslant m, k \leqslant j\} \tag{2.3}$$

can be used as a basis of $L^2(\mathrm{SU}(2))$, where the special functions D^j_{mk} are the usual matrix elements of the unirreps of SU(2) (Rose 1957).

The definition of the left- and right-regular representation (Gel'fand *et al* 1966, Coleman 1968) allows us to use $L^2(\mathrm{SU}(2))$ in a natural way as carrier space for a unitary representation U of the direct product group $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

$$\begin{aligned} &U\colon (\omega_1, \omega_2) \to U(\omega_1, \omega_2) \\ &[U(\omega_1, \omega_2) f](\omega) = f(\omega_1^{-1} \omega \omega_2). \end{aligned} \tag{2.4}$$

However, this representation U is not a faithful one. U is only isomorphic to SO(4, R). By means of the properties of the unirreps we obtain for

$$U(\omega_1, \omega_2)\mathscr{D}^j_{mk} = \sum_{m'k'} D^j_{m'm}(\omega_1)D^{j*}_{k'k}(\omega_2)\mathscr{D}^j_{m'k'} \tag{2.5}$$

i.e. the elements $\mathscr{D}^j_{mk}$, $-j\leq m,\ k\leq j$ transform according to the unirrep $D^j(\omega_1)\otimes D^{j*}(\omega_2)$ of SU(2)×SU(2). Because of the equivalence

$$D^{j*}(\omega)\simeq D^j(\omega) \qquad \text{for all } j \tag{2.6}$$

$$D^{j*}(\omega) = U^j D^j(\omega) U^{j+}, \qquad U^j_{mk} = (-1)^{j+k}\delta_{m,-k} \tag{2.7}$$

we introduce, instead of (2.3),

$$Q^j_{mk} = \sum_{k'} \mathscr{D}^j_{mk'} U^j_{k'k} = (2j+1)^{1/2}(-1)^{j+k} D^{j*}_{m,-k} \tag{2.8}$$

the elements of another basis of L^2(SU(2)) being already SU(2)×SU(2)-adapted since they transform according to the unirreps $D^{jj}(\omega_1, \omega_2) = D^j(\omega_1)\otimes D^j(\omega_2)$:

$$U(\omega_1, \omega_2)Q^j_{mk} = \sum_{m'k'} D^j_{m'm}(\omega_1)D^j_{k'k}(\omega_2)Q^j_{m'k'}. \tag{2.9}$$

Therefore L^2(SU(2)) decomposes into a direct sum of the unirreps $D^{jj}(\omega_1, \omega_2)$ $j = 0, \frac{1}{2}, 1, \ldots$ under the action of the unitary representation U of SU(2)×SU(2). Each unirrep $D^{jj}(\omega_1, \omega_2)$ occurs in U only once (Coleman 1968).

Now we introduce the left- ($^{(L)}\mathscr{A}$(SU(2))) and the right-group algebra ($^{(R)}\mathscr{A}$(SU(2))) for the definition of some special IT with respect to SU(2)×SU(2). The elements of $^{(i)}\mathscr{A}$(SU(2)) (i = left and right) are given by ($\omega_0 = (0, 0, 0)$):

$$^{(L)}A = \int d\mu(\omega_1)a(\omega_1)U(\omega_1, \omega_0), \qquad a\in L^2(SU(2)) \tag{2.10}$$

$$^{(R)}B = \int d\mu(\omega_2)b(\omega_2)U(\omega_0, \omega_2), \qquad b\in L^2(SU(2)). \tag{2.11}$$

The definition and the properties of the so called 'units' which are elements of a basis of $^{(i)}\mathscr{A}$(SU(2)) are well known (Naimark 1964):

$$^{(L)}E^j_{mk} = (2j+1)\int d\mu(\omega_1)D^{j*}_{mk}(\omega_1)U(\omega_1, \omega_0) \tag{2.12}$$

$$^{(R)}E^{j'}_{m'k'} = (2j'+1)\int d\mu(\omega_2)D^{j'*}_{m'k'}(\omega_2)U(\omega_0, \omega_2) \tag{2.13}$$

$$^{(i)}E^{j+}_{mk} = {}^{(i)}E^j_{km} \tag{2.14}$$

$$^{(i)}E^j_{mk}\,{}^{(i)}E^{j''}_{m''k''} = \delta_{jj''}\delta_{km''}\,{}^{(i)}E^j_{mk''} \tag{2.15}$$

$$U(\omega_1, \omega_2)^{(i)}E^j_{mk} = \sum_{m'} D^j_{m'm}(\omega_i)^{(i)}E^j_{m'k} \tag{2.16}$$

$$U(\omega_0, \omega_0) = \sum_{jm}{}^{(i)}E^j_{mm} \qquad (i = \text{left or right}). \tag{2.17}$$

Therefore the operators

$$^{(L)}E^j_{mm'}\,{}^{(R)}E^{j'}_{kk'} = {}^{(R)}E^{j'}_{kk'}\,{}^{(L)}E^j_{mm'} = E^{jj'}_{mk,m'k'} \tag{2.18}$$

which are composed of the commuting operators (2.12, 13) can be seen as the representation of the units ${}^{(i)}E^{jj'}_{mk,m'k'}$ (i = left or right) of the analogously defined group algebra ${}^{(i)}\mathscr{A}(\mathrm{SU}(2)\times\mathrm{SU}(2))$. However, only the unirreps $D^{jj}(\omega_1, \omega_2)$ are contained in U, the operators (2.18) are only different from the zero operator if and only if $j'=j$:

$$E^{jj}_{mk,m'k'} = {}^{(L)}E^{j}_{mm'}\,{}^{(R)}E^{j}_{kk'} \tag{2.19}$$

$${}^{(L)}E^{j}_{mm'}Q^{j_2}_{m_2k_2} = \delta_{jj_2}\delta_{m'm_2}Q^{j_2}_{mk_2} \tag{2.20}$$

$${}^{(R)}E^{j'}_{kk'}Q^{j_2}_{m_2k_2} = \delta_{j'j_2}\delta_{k'k_2}Q^{j_2}_{m_2k}. \tag{2.21}$$

3. IT with respect to SU(2) × SU(2)

According to the general definition (Lomont 1959) we call the set of operators

$$\{T^{AB}_{ab}:\ -A\leqslant a\leqslant A,\ -B\leqslant b\leqslant B\} \tag{3.1}$$

an IT of rank A, B with respect to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ if its components transform according to the unirrep $D^{AB}(\omega_1, \omega_2) = D^{A}(\omega_1)\otimes D^{B}(\omega_2)$:

$$U(\omega_1, \omega_2)T^{AB}_{ab}U^{+}(\omega_1, \omega_2) = \sum_{a'b'} D^{A}_{a'a}(\omega_1)D^{B}_{b'b}(\omega_2)T^{AB}_{a'b'}. \tag{3.2}$$

We suppose that the product of the operators on the left-hand side of equation (3.2) and the linear combinations on the right-hand side of equation (3.2) have a meaning in the usual sense as products and sums of operators in the Hilbert space $L^2(\mathrm{SU}(2))$. An equivalent definition can be given by means of the elements of the left- and right-Lie algebras of SU(2):

$$[{}^{(L)}J_{\pm}, T^{AB}_{ab}] = [(A\mp a)(A\pm a+1)]^{1/2}T^{AB}_{a\pm1,b} \tag{3.3}$$

$$[{}^{(L)}J_3, T^{AB}_{ab}] = aT^{AB}_{ab} \tag{3.4}$$

$$[{}^{(R)}J_{\pm}, T^{AB}_{ab}] = [(B\mp b)(B\pm b+1)]^{1/2}T^{AB}_{a,b\pm1} \tag{3.5}$$

$$[{}^{(R)}J_3, T^{AB}_{ab}] = bT^{AB}_{ab}. \tag{3.6}$$

Of course because of $[U(\omega_1, \omega_0), U(\omega_0, \omega_2)] = 0$ the elements of the left- and right-Lie algebras commute. They are given by

$$[{}^{(L)}J_{\pm}f](\omega) = \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\alpha}\left(\cot\beta\,\partial_\alpha \mp \mathrm{i}\partial_\beta - \frac{1}{\sin\beta}\partial_\gamma\right)f(\omega) \tag{3.7}$$

$$[{}^{(L)}J_3f](\omega) = -\mathrm{i}\partial_\alpha f(\omega) \tag{3.8}$$

$$[{}^{(R)}J_{\pm}f](\omega) = \mathrm{i}\mathrm{e}^{\mp\mathrm{i}\gamma}\left(\cot\beta\,\partial_\gamma \pm \mathrm{i}\partial_\beta - \frac{1}{\sin\beta}\partial_\alpha\right)f(\omega) \tag{3.9}$$

$$[{}^{(R)}J_3f](\omega) = +\mathrm{i}\partial_\gamma f(\omega). \tag{3.10}$$

In order to answer the question: which ranks A, B of IT can be realized in $L^2(\mathrm{SU}(2))$, it suffices to remember that U is isomorphic to SO(4, R) or to investigate the Wigner–Eckart theorem for the group $\mathrm{SU}(2)\times\mathrm{SU}(2)$:

$$\langle Q^{j_1}_{m_1k_1}, T^{AB}_{ab}Q^{j_2}_{m_2k_2}\rangle = (Aa, j_2m_2|j_1m_1)(Bb, j_2k_2|j_1k_1)(j_1\|T^{AB}\|j_2). \tag{3.11}$$

The matrix elements (3.11) are different from zero in principle if

$$|j_1-j_2|\leqslant A, B\leqslant j_1+j_2 \tag{3.12}$$

is satisfied. This has as a consequence that the pair A, B can take either the values

$$A=0,1,2,\dots \quad \text{and} \quad B=0,1,2,\dots \tag{3.13}$$

or

$$A=\tfrac{1}{2},\tfrac{3}{2},\dots \quad \text{and} \quad B=\tfrac{1}{2},\tfrac{3}{2},\dots . \tag{3.14}$$

The problem of decomposing a given operator $\mathcal{O}$ in IT components with respect to SU(2)×SU(2) can be done in the following way (Dirl and Kasperkovitz 1976, Dirl 1974a,b):

$$\mathcal{O}=\sum_{\substack{AB\\ab}} T^{AB}_{ab,ab}[\mathcal{O}] \tag{3.15}$$

$$T^{AB}_{ab,a'b'}[\mathcal{O}]=(2A+1)(2B+1)$$

$$\times\int\int \mathrm{d}\mu(\omega_1)\,\mathrm{d}\mu(\omega_2) D^{A*}_{aa'}(\omega_1) D^{B*}_{bb'}(\omega_2) U(\omega_1,\omega_2)\mathcal{O}U^{+}(\omega_1,\omega_2). \tag{3.16}$$

We assume just as before that the right-hand sides of equations (3.15, 16) are meaningful in the usual sense and that the operator $\mathcal{O}$ belongs to the class of operators whose domain of definition contains (at least) the elements of the basis (2.8).

In the following sections we introduce mainly two different types of IT which are closely related to the representation theory of the group SU(2), and we try to define with them new ones which are sufficient to express every IT component as a unique linear combination of them.

4. IT within the group algebra

The first type of IT which we introduce consists of elements of the tensor basis of ${}^{(i)}\mathcal{A}(\mathrm{SU}(2))$ (Dirl and Kasperkovitz 1976, Kasperkovitz and Dirl 1974). They are defined with the matrix elements $U^{j}_{mm'}$ (see equation (2.7)) and the Clebsch–Gordan coefficients (CG coefficients) of SU(2)

$${}^{(\mathrm{L})}T^{j;J}_{M}=\sum_{k}(-1)^{j+k}(jM+k,j-k|JM)\,{}^{(\mathrm{L})}E^{j}_{M+k,k} \tag{4.1}$$

$${}^{(\mathrm{R})}T^{j';J'}_{K}=\sum_{k'}(-1)^{j'+k'}(j'\,K+k',j'-k'|J'K)\,{}^{(\mathrm{R})}E^{j'}_{K+k',k'} \tag{4.2}$$

$${}^{(\mathrm{L})}E^{j}_{mk}=(-1)^{j+k}\sum_{J}(jm,j-k|J\,m-k)\,{}^{(\mathrm{L})}T^{j;J}_{m-k} \tag{4.3}$$

$${}^{(\mathrm{R})}E^{j'}_{m'k'}=(-1)^{j'+k'}\sum_{J'}(j'm',j'-k'|J'\,m'-k')\,{}^{(\mathrm{R})}T^{j';J'}_{m'-k'}. \tag{4.4}$$

The following sets of operators:

$$\{{}^{(\mathrm{L})}T^{j;J}_{M}:\ -J\leqslant M\leqslant J\} \tag{4.5}$$

$$\{{}^{(\mathrm{R})}T^{j';J'}_{K}:\ -J'\leqslant K\leqslant J'\} \tag{4.6}$$

are IT of rank J, 0 and 0, J' with respect to SU(2)×SU(2). Because of equation (2.19) we have

$${}^{(\mathrm{L})}T^{j;J}_{M}\,{}^{(\mathrm{R})}T^{j';J'}_{K}=\delta_{jj'}\,{}^{(\mathrm{L})}T^{j;J}_{M}\,{}^{(\mathrm{R})}T^{j;J'}_{K} \tag{4.7}$$

and each set of the operators

$$\{{}^{(L)}T_M^{j;J}\,{}^{(R)}T_K^{j;J'}: -J \leq M \leq J,\ -J' \leq K \leq J'\} \tag{4.8}$$

forms an IT of rank J, J'. The following properties of the IT components ${}^{(i)}T_M^{j;J}$ (i = left and right) are of importance if one tries to define some new ones

$$ {}^{(i)}T_M^{j;J+} = (-1)^{M\,(i)}T_{-M}^{j;J} \tag{4.9}$$

$$ {}^{(i)}T_M^{j;J\,(i)}T_K^{j;J'} = (-1)^{2j+J}[(2J+1)(2J'+1)]^{1/2} \sum_{J''} W(JjJ''j, jJ')(JM, J'K|J''M+K)\,{}^{(i)}T_{M+K}^{j;J''}, \tag{4.10}$$

where the coefficients $W(abcd, ef)$ are the usual Racah coefficients (Rose 1957). When calculating the matrix elements of the IT components (4.1, 2, 8) one has to use equations (2.20, 21), (3.11) and the well known symmetry relations of the CG coefficients of SU(2) (Rose 1957).

5. Matrix elements of the unirreps of SU(2) as IT

The second type of IT which we introduce is closely related to the matrix elements of the unirreps of SU(2) (Dirl and Kasperkovitz 1976, Judd and Vogel 1975):

$$[Q_{MK}^{RR}f](\omega) = Q_{MK}^{R}(\omega)f(\omega) \qquad \text{for all } f \in L^2(\mathrm{SU}(2)) \tag{5.1}$$

$$U(\omega_1, \omega_2)Q_{MK}^{RR}U^{+}(\omega_1, \omega_2) = \sum_{M'K'} D_{M'M}^{R}(\omega_1)D_{K'K}^{R}(\omega_2)Q_{M'K'}^{RR}. \tag{5.2}$$

These IT components of rank R, R have properties resulting from the properties (Rose 1957) of the corresponding matrix elements of the unirreps of SU(2):

$$Q_{MK}^{RR+} = (-1)^{2R+M+K}Q_{-M,-K}^{RR} \tag{5.3}$$

$$Q_{MK}^{RR}Q_{M'K'}^{R'R'} = \sum_{R''}\left(\frac{(2R+1)(2R'+1)}{2R''+1}\right)^{1/2}(RM, R'M'|R''M+M')$$
$$\times(RK, R'K'|R''K+K')Q_{M+M',K+K'}^{R''R''} \tag{5.4}$$

$$\sum_{M} Q_{MK}^{RR+}Q_{MK'}^{RR} = \sum_{M} Q_{KM}^{RR+}Q_{K'M}^{RR} = (2R+1)\delta_{KK'}. \tag{5.5}$$

When calculating the matrix elements of the IT components (5.1) one has to use the following relation:

$$Q_{MK}^{RR}Q_{mk}^{j} = \sum_{j''}\left(\frac{(2R+1)(2j+1)}{2j''+1}\right)^{1/2}(RM, jm|j''M+m)(RK, jk|j''K+k)Q_{M+m,K+k}^{j''}. \tag{5.6}$$

6. Correlations between special IT

The special IT (4.1, 2) and (5.1) allow us to construct much more general ones. The following list shows that one has a very large number of possibilities to define IT with respect to SU(2)×SU(2):

$$ {}^{j}T_{ab}^{(\underline{B}J)AB}=\sum_{M}(B\,a-M,JM|Aa)Q_{a-M,b}^{\underline{B}B}\,{}^{(L)}T_{M}^{j;J} \tag{6.1}$$

$$ {}^{j}T_{ab}^{(J\underline{B})AB}=\sum_{M}(J\,a-M,BM|Aa)\,{}^{(L)}T_{a-M}^{j;J}Q_{Mb}^{BB} \tag{6.2}$$

$$ {}^{j'}T_{ab}^{\{\underline{A}J'\}AB}=\sum_{K}(A\,b-K,J'K|Bb)Q_{a,b-K}^{AA}\,{}^{(R)}T_{K}^{j',J'} \tag{6.3}$$

$$ {}^{j'}T_{ab}^{\{J'\underline{A}\}AB}=\sum_{K}(J'\,b-K,AK|Bb)\,{}^{(R)}T_{b-K}^{j';J'}Q_{aK}^{AA} \tag{6.4}$$

$$ {}^{jj'}T_{ab}^{(\underline{R}JJ')AB}=\sum_{MK}(R\,a-M,JM|Aa)(R\,b-K,J'K|Bb)Q_{a-M,b-K}^{RR}\,{}^{(L)}T_{M}^{j;J}\,{}^{(R)}T_{K}^{j;J'} \tag{6.5}$$

$$ {}^{jj'}T_{ab}^{(JJ'\underline{R})AB}=\sum_{MK}(J\,a-M,RM|Aa)(J'\,b-K,RK|Bb)\,{}^{(L)}T_{a-M}^{j;J}\,{}^{(R)}T_{b-K}^{j;J'}Q_{MK}^{RR} \tag{6.6}$$

$$ {}^{jj'}T_{ab}^{(J\underline{R}J')AB}=\sum_{MK}(J\,a-M,RM|Aa)(R\,b-K,J'K|Bb)\,{}^{(L)}T_{a-M}^{j;J}Q_{M,b-K}^{RR}\,{}^{(R)}T_{K}^{j';J'} \tag{6.7}$$

$$ {}^{jj'}\hat{T}_{ab}^{(J\underline{R}J')AB}=\sum_{MK}(J\,b-K,RK|Bb)(R\,a-M,J'M|Aa)\,{}^{(R)}T_{b-K}^{j;J}Q_{a-M,K}^{RR}\,{}^{(L)}T_{M}^{j';J'}. \tag{6.8}$$

Now it is obvious that the operators (6.1–8) are IT components of rank A, B. However, one cannot expect that these operators are all linearly independent respective of the fact that there do not exist correlations between them. In fact if one investigates the following operators one obtains the following results.

$$ {}^{j}T_{0b}^{(\underline{J}J)0J}=\sum_{M}(J-M,JM|00)Q_{-M,b}^{JJ}\,{}^{(L)}T_{M}^{j;J}={}^{(R)}T_{b}^{j;J} \tag{6.9}$$

$$ {}^{j}T_{a0}^{\{\underline{J}J\}J0}=\sum_{K}(J-K,JK|00)Q_{a,-K}^{JJ}\,{}^{(R)}T_{K}^{j;J}={}^{(L)}T_{a}^{j;J} \tag{6.10}$$

$$ {}^{j}T_{0b}^{(J\underline{J})0J}=\sum_{M}(J-M,JM|00)\,{}^{(L)}T_{-M}^{j;J}Q_{Mb}^{JJ}={}^{(R)}T_{b}^{j;J} \tag{6.11}$$

$$ {}^{j}T_{a0}^{\{J\underline{J}\}J0}=\sum_{K}(J-K,JK|00)\,{}^{(R)}T_{-K}^{j;J}Q_{aK}^{JJ}={}^{(L)}T_{a}^{j;J}. \tag{6.12}$$

To prove these relations one has to use equations (6.3, 5) and note that J must be an integer. Now, inserting equation (6.10) in equation (6.1) or equation (6.12) in equation (6.2) we obtain for

$$ {}^{j}T_{ab}^{(\underline{B}J)AB}=(-1)^{J+A-B}\,{}^{j}T_{ab}^{\{\underline{A}J\}AB} \tag{6.13}$$

$$ {}^{j}T_{ab}^{(J\underline{B})AB}=(-1)^{J+A-B}\,{}^{j}T_{ab}^{\{J\underline{A}\}AB} \tag{6.14}$$

where we have used equation (5.4), the decompositions (use equations (3.15, 16))

$$ Q_{a,b-K}^{AA}\,{}^{(R)}T_{K}^{j;J}=\sum_{H}(A\,b-K,JK|Hb)\,{}^{j}T_{ab}^{\{\underline{A}J\}AH} \tag{6.15}$$

$$ {}^{(R)}T_{-K}^{j;J}Q_{a,b+K}^{AA}=\sum_{H}(J-K,A\,b+K|Hb)\,{}^{j}T_{ab}^{\{J\underline{A}\}AH} \tag{6.16}$$

and the definition of the Racah coefficients (Rose 1957). (Note that equations (6.9–12) are contained as special cases.) After a straightforward calculation by using equations

(6.10, 15), (5.4) one obtains

$$ {}^{jj}T^{(RJJ')AB}_{ab} = (-1)^{2j-J'}[(2R+1)(2J+1)(2J'+1)]^{1/2} \times \sum_{J''} (-1)^{J''}(2J''+1)^{1/2} W(JjJ''j, jJ') W(RJBJ'', AJ')^{j}T^{\{AJ''\}AB}_{ab} \tag{6.17} $$

where the operators (6.5) are immediately expressible in terms of the IT components (6.1) by means of the relation (6.13). Clearly in the same way the operators (6.6) can be evaluated in terms of the operators (6.2) or (6.4) by an analogous formula. Finally one can expect that there must exist a relation between the operators (6.7) and (6.8).

$$ {}^{jj'}T^{(JRJ')AB}_{ab} = (-1)^{J+A-R}[(2R+1)(2J'+1)]^{1/2} \sum_{R''} (2R''+1)^{1/2} W(JABJ', RR'')^{jj'}\hat{T}^{(JR''J')AB}_{ab}. \tag{6.18} $$

For the proof of this relation one has to use the decomposition

$$ {}^{(R)}T^{j;J}_{-K} Q^{R''R''}_{a-M,b+K} {}^{(L)}T^{j';J}_{M} = \sum_{A''B''} (R''\, a-M, J'M|A''a)(J\, -K, R''\, b+K|B''b)^{jj'}\hat{T}^{(JR''J')A''B''}_{ab} \tag{6.19} $$

and equation (5.4). Because of the linear dependences (6.13, 14, 17, 18) we can assume that at most the operators (6.2) and (6.3) are linearly independent and probably that the last ones are expressible as linear combinations of the operators (6.7).

When calculating the matrix elements of the operators (6.3, 2, 7) we need the following formulae:

$$ {}^{j}T^{\{AJ\}AB}_{ab} Q^{j_2}_{m_2k_2} = \delta_{jj_2}(-1)^{2j+A+B+J}[(2A+1)(2B+1)(2J+1)(2j+1)]^{1/2} \times \sum_{j''} (Aa, jm_2|j''a+m_2)(Bb, jk_2|j''b+k_2) \frac{W(BJj''j, Aj)}{\sqrt{2j''+1}} Q^{j''}_{a+m_2,b+k_2} \tag{6.20} $$

$$ {}^{j}T^{(JB)AB}_{ab} Q^{j_2}_{m_2k_2} = (-1)^{j+j_2-2A+J}\left(\frac{(2A+1)(2B+1)(2J+1)(2j_2+1)}{2j+1}\right)^{1/2} \times (Aa, j_2m_2|j\, a+m_2)(Bb, j_2k_2|j\, b+k_2) W(BJj_2j, Aj) Q^{j}_{a+m_2,b+k_2} \tag{6.21} $$

$$ {}^{jj'}T^{(JRJ')AB}_{ab} Q^{j_2}_{m_2k_2} = \delta_{j'j_2}(-1)^{A+B}[(2A+1)(2B+1)(2J+1)(2J'+1)]^{1/2} \times (Aa, j'm_2|j\, a+m_2)(Bb, j'k_2|j\, b+k_2) \times W(ARjj, Jj') W(BRj'j', J'j) Q^{j}_{a+m_2,b+k_2}. \tag{6.22} $$

Consequently the reduced matrix elements of these three operators are given by

$$ (j_1\|{}^{j}T^{\{AJ\}AB}\|j_2) = \delta_{jj_2}(-1)^{2j+A+B+J}\left(\frac{(2A+1)(2B+1)(2J+1)(2j+1)}{2j_1+1}\right)^{1/2} W(BJj_1j, Aj) \tag{6.23} $$

$$ (j_1\|{}^{j}T^{(JB)AB}\|j_2) = \delta_{jj_1}(-1)^{j+j_2-2A+J}\left(\frac{(2A+1)(2B+1)(2J+1)(2j_2+1)}{2j+1}\right)^{1/2} W(BJj_2j, Aj) \tag{6.24} $$

$$ (j_1\|{}^{jj'}T^{(JRJ')AB}\|j_2) = \delta_{jj_1}\delta_{j'j_2}(-1)^{A+B}[(2A+1)(2B+1)(2J+1)(2J'+1)]^{1/2} \times W(ARjj, Jj') W(BRj'j', J'j). \tag{6.25} $$

7. Complete sets of special IT

The term 'completeness of a set of IT' already used in § 6 refers to the following linear space of operators:

$$V_Q=\{\mathcal{O}: \mathcal{D}_{\mathcal{O}}\supset\{Q^j_{mk}\}\}. \tag{7.1}$$

(The symbol $\mathcal{D}_{\mathcal{O}}$ denotes the domain of definition of the operator $\mathcal{O}$.) This means that we define a set of IT (which are composed only of the operators (4.1, 2) and (5.1)) so that every operator $\mathcal{O}\in V_Q$ can be expressed as a unique linear combination of these IT. It is obvious that just the operators (6.7) have this property. However, one must be aware that the operators (6.7) for a given pair j, j' are only different from zero if

$$\begin{aligned} &|j-j'|\leqslant R, A, B\leqslant j+j' \\ &|R-J|\leqslant A\leqslant R+J, \qquad 0\leqslant J\leqslant 2j \\ &|R-J'|\leqslant B\leqslant R+J', \qquad 0\leqslant J'\leqslant 2j' \end{aligned} \tag{7.2}$$

are satisfied and those operators which are different from the zero operator are linearly dependent for fixed A, B if R, J, J' varies. The last assertion follows immediately from equation (6.25). However, to be sure that for a fixed j, j' all the allowed A, B can be realized we choose for the quantities J, R, J' the following values:

$$J=2j, \qquad R=j+j', \qquad J'=2j'. \tag{7.3}$$

Therefore the desired expansions are of the form

$$\mathcal{O}=\sum_{\substack{AB\\ab}} T^{AB}_{ab,ab}[\mathcal{O}] \tag{7.4}$$

$$T^{AB}_{ab,ab}[\mathcal{O}]=\sum_{jj'} c^{jj'(AB)}_{ab}\, {}^{jj'}T^{(JRJ')AB}_{ab} \tag{7.5}$$

where the equation (7.3) has to be taken into account. The coefficients $c^{jj'(AB)}_{ab}$ appearing in the expansion (7.5) are given by (use equation (6.25))

$$(j\|T^{AB}_{ab}[\mathcal{O}]\|j')=(j\|^{jj'}T^{(2j,j+j',2j')AB}\|j')c^{jj'(AB)}_{ab}. \tag{7.6}$$

8. Special IT composed of Q^{RR}_{MK} and elements of the enveloping algebra

It is well known (Rose 1957) that the elements

$${}^{(i)}J_{\pm1}=\mp\frac{1}{\sqrt{2}}{}^{(i)}J_{\pm}, \qquad {}^{(i)}J_0={}^{(i)}J_3 \tag{8.1}$$

of the left- or right-Lie algebra of SU(2) are IT components of rank 1, 0 or 0, 1 with respect to SU(2)×SU(2). As suggested by the relations (6.9–12) one can expect similar relations for the operators (8.1). Indeed this is easily proved by direct calculation and we get the relations (Marshalek 1975)

$${}^{(R)}J_K=\sum_M (1-M, 1M|00)Q^{11}_{-M,K}\,{}^{(L)}J_M=\sum_M (1-M, 1M|00)\,{}^{(L)}J_{-M}Q^{11}_{MK} \tag{8.2}$$

$${}^{(L)}J_M=\sum_K (1-K, 1K|00)Q^{11}_{M,-K}\,{}^{(R)}J_K=\sum_K (1-K, 1K|00)\,{}^{(R)}J_{-K}Q^{11}_{MK}. \tag{8.3}$$

Furthermore it is well known that the operators (8.1) can be used (by means of equations (3.15, 16)) for the definition of IT of rank J, 0 respectively 0, J' which belong to the enveloping algebra. Denoting the IT components of such IT of rank J, 0 by

$$^{(L)}Z^J_M = \{\text{power series in } {}^{(L)}J_{M'}\} \tag{8.4}$$

it is obvious that analogously to equation (6.9) one can use the relation

$$^{(R)}Z^J_K = \sum_M (J-M, JM|00) Q^{JJ}_{-M,K} {}^{(L)}Z^J_M \tag{8.5}$$

for the definition of the corresponding IT components of an IT of rank 0, J. Of course such relations can be of some interest for practical calculations since it suffices to construct the IT components (8.4) and by means of equation (8.5) obtain immediately the other IT. We realize that these operators are related to the special IT components (4.1, 2) in the following way:

$$^{(L)}Z^J_M = \sum_j c^{j,J}\, {}^{(L)}T^{j;J}_M \tag{8.6}$$

$$^{(R)}Z^{J'}_K = \sum_j c^{j,J'}\, {}^{(R)}T^{j;J'}_K. \tag{8.7}$$

The coefficients $c^{j,J}$ are given as the quotient of the corresponding reduced matrix elements. For example, in the special case $J=1$ we obtain, because of

$$^{(L)}J_M Q^j_{mk} = -[j(j+1)]^{1/2}(1M, jm|jM+m) Q^j_{M+m,k} \tag{8.8}$$

$$^{(L)}T^{j';1}_M Q^j_{mk} = \delta_{jj'}(-1)^{2j-1}\left(\frac{3}{2j+1}\right)^{1/2}(1M, jm|j\,M+m) Q^j_{M+m,k} \tag{8.9}$$

for the coefficients

$$c^{j,1} = (-1)^{2j}\left(\frac{j(j+1)(2j+1)}{3}\right)^{1/2} \tag{8.10}$$

(For the operators $^{(R)}J_K$ and $^{(R)}T^{j;1}_K$ if we apply them to the elements Q^j_{mk} the relations are exactly the same as (8.8, 9).) Finally the operators

$$^{(L)}Z^J_M {}^{(R)}Z^{J'}_K = {}^{(R)}Z^{J'}_K {}^{(L)}Z^J_M = Z^{JJ'}_{MK} \tag{8.11}$$

are IT components of an IT of rank J, J'.

According to similar arguments stated in § 6 only the operators of the type

$$Z^{(\underline{A}J)AB}_{ab} = \sum_K (A\,b-K, JK|Bb) Q^{AA}_{a,b-K} {}^{(R)}Z^J_K \tag{8.12}$$

$$Z^{(J\underline{B})AB}_{ab} = \sum_M (J\,a-M, BM|Aa)\, {}^{(L)}Z^J_{a-M} Q^{BB}_{Mb} \tag{8.13}$$

$$Z^{(J\underline{R}J')AB}_{ab} = \sum_{MK} (J\,a-M, RM|Aa)(R\,b-K, J'K|Bb)\, {}^{(L)}Z^J_{a-M} Q^{RR}_{M,b-K} {}^{(R)}Z^{J'}_K \tag{8.14}$$

can be of importance because one can show that there exist relations analogous to (6.13, 14, 17, 18). However, one must be aware that even the operators (8.14) cannot be complete in the sense of equation (7.5). This means that the operators (8.12–14) can only be complete for a certain class of operators.

Therefore as a consequence of the preceding considerations, an expansion of the type

$$\mathcal{O}=\sum_{\substack{AB\\ab}} c_{ab}^{AB} Z_{ab}^{\{\underline{A}J\}AB} \qquad (J \text{ fixed}) \tag{8.15}$$

(which was in principle discussed by Beers and Millman 1975, with $J=1$) holds only for operators whose reduced matrix elements are proportional to those of the operators (8.12):

$$(j_1|T_{ab}^{AB}[\mathcal{O}]\|j_2)=\text{multiple of } (j_1\|Z^{\{\underline{A}J\}AB}\|j_2) \qquad \text{for all } j_1, j_2 \text{ and } A, B \tag{8.16}$$

and where the IT components $T_{ab,ab}^{AB}[\mathcal{O}]$ are only different from zero if

$$|A-J|\leqslant B\leqslant A+J \tag{8.17}$$

is satisfied. The calculation of the coefficients c_{ab}^{AB} can be somewhat simplified if one uses (3.16) and (5.5):

$$\sum_{a''} Q_{a'',A}^{AA+} T_{a''B,ab}^{AB}[\mathcal{O}]=c_{ab}^{AB}(2A+1)(AA, JB-A|BB)^{(\mathrm{R})}Z_{B-A}^{J}. \tag{8.18}$$

(Further simplifications arise if one takes the special values $A=B$ if $A\geqslant B$ or $B=A$ if $A\leqslant B$.) However, one must not forget that the ranks of the IT components occurring in equation (8.15) must satisfy the conditions

$$A=0, \tfrac{1}{2}, 1, \ldots \qquad \text{and} \qquad |A-J|\leqslant B\leqslant A+J \qquad (J \text{ fixed}). \tag{8.19}$$

Finally in order to discuss if it is possible to find operators which are linear combinations of IT of the type (8.12–14) with properties similar to creation and destruction operators, we discuss IT of rank $\frac{1}{2}, \frac{1}{2}$. For this purpose we need (cf equation (8.8)) also

$$^{(R)}J_K Q_{mk}^{j}=-[j(j+1)]^{1/2}(1K, jk|jK+k)Q_{m,K+k}^{j}. \tag{8.20}$$

We obtain by utilizing (8.8, 21), (5.6) and the definition of the Racah coefficients for

$$Z_{ab}^{\{\underline{A}1\}AB}Q_{mk}^{j}=(-1)^{B}(2j+1)[j(j+1)(2A+1)(2B+1)]^{1/2}$$
$$\times\sum_{j''}(-1)^{j-j''}\frac{W(ABjj, 1j'')}{(2j''+1)^{1/2}}(Aa, jm|j''\, a+m)(Bb, jk|j''\, b+k)Q_{a+m,b+k}^{j''} \tag{8.21}$$

$$Z_{ab}^{(1\underline{B})AB}Q_{mk}^{j}=(-1)^{A}[(2j+1)(2A+1)(2B+1)]^{1/2}\sum_{j''}(-1)^{j''-j}[j''(j''+1)]^{1/2}$$
$$\times W(ABj''j'', 1j)(Aa, jm|j''\, a+m)(Bb, jk|j''\, b+k)Q_{a+m,b+k}^{j''}. \tag{8.22}$$

Specializing these results for $A=B=\frac{1}{2}$, we get

$$Z_{ab}^{\{\underline{\frac{1}{2}}1\}\frac{1}{2}\frac{1}{2}}Q_{mk}^{j}=j\left(\frac{2j+1}{3(j+1)}\right)^{1/2}(\tfrac{1}{2}a, jm|j+\tfrac{1}{2}, a+m)(\tfrac{1}{2}b, jk|j+\tfrac{1}{2}, b+k)\, Q_{a+m,b+k}^{j+\frac{1}{2}}$$
$$-(j+1)\left(\frac{2j+1}{3j}\right)^{1/2}(\tfrac{1}{2}a, jm|j-\tfrac{1}{2}, a+m)(\tfrac{1}{2}b, jk|j-\tfrac{1}{2}, b+k)\, Q_{a+m,b+k}^{j-\frac{1}{2}} \tag{8.23}$$

$$Z_{ab}^{(1\underline{\frac{1}{2}})\frac{1}{2}\frac{1}{2}}Q_{mk}^{j}=-(Z_{ab}^{\{\underline{\frac{1}{2}}1\}\frac{1}{2}\frac{1}{2}}+\tfrac{1}{2}\sqrt{3}\,Q_{ab}^{\frac{1}{2}\frac{1}{2}})Q_{mk}^{j}. \tag{8.24}$$

And for the third type of IT components we have manufactured for

$$Z_{ab}^{(1\underline{R}1)AB}Q^{j}_{mk}=(-1)^{A+B}[(2A+1)(2B+1)j(j+1)(2j+1)]^{1/2}$$

$$\times\sum_{j''}(-1)^{j''-j}[j''(j''+1)(2j''+1)]^{1/2}W(ARj''j'',1j)$$

$$\times W(BRjj,1j'')(Aa,jm|j''\,a+m)(Bb,jk|j''\,b+k)Q^{j''}_{a+m,b+k} \qquad (8.25)$$

and therefore for the special case $R=\frac{1}{2}$, $A=B=\frac{1}{2}$

$$Z_{ab}^{(1\frac{1}{2}1)\frac{1}{2}\frac{1}{2}}Q^{j}_{mk}= -\tfrac{1}{3}ij(j+\tfrac{3}{2})(\tfrac{1}{2}a,jm|j+\tfrac{1}{2},a+m)(\tfrac{1}{2}b,jk|j+\tfrac{1}{2},b+k)\,Q^{j+\frac{1}{2}}_{a+m,b+k}$$

$$+\tfrac{1}{3}i(j+1)(j-\tfrac{1}{2})(\tfrac{1}{2}a,jm|j-\tfrac{1}{2},a+m)(\tfrac{1}{2}b,jk|j-\tfrac{1}{2},b+k)Q^{j-\frac{1}{2}}_{a+m,b+k} \qquad (8.26)$$

and the special case $R=\frac{3}{2}$, $A=B=\frac{1}{2}$

$$Z_{ab}^{(1\frac{3}{2}1)\frac{1}{2}\frac{1}{2}}Q^{j}_{mk}=-Z_{ab}^{(1\frac{1}{2}1)\frac{1}{2}\frac{1}{2}}Q^{j}_{mk}. \qquad (8.27)$$

This shows that we cannot introduce such IT of rank $\frac{1}{2},\frac{1}{2}$ (which are linear combinations of the operators (8.12–14)) where the operators (8.1) occur linearly, which raises (or lowers) the indices $j\to j+\frac{1}{2}(-\frac{1}{2})$, $m\to m+a$, $k\to k+b$ of the elements (2.8). (We can define such operators only as linear combinations of IT components of the type (6.7).)

9. Conclusions

It was the aim of this paper to show by means of the example $L^2(SU(2))$ how to construct, from simple but convenient operators, IT with respect to $SU(2)\times SU(2)$ using this Hilbert space as the representation space. However, the construction of such IT presupposes knowledge of the unirreps, the unitary matrices U relating the unirreps to their complex conjugates and the CG coefficients for SU(2). In the case of the group $SU(2)\times SU(2)$ we are able:

(i) to construct a complete set of IT whose elements are composed of operators which are closely related to the representation theory of the group SU(2);

(ii) to show operator identities and linear dependences between special IT, which give rise to correlation of Racah coefficients (compare equation (6.18) with equation (6.26)); and

(iii) to construct special IT (composed of IT which are closely related to the matrix elements of the unirreps of SU(2) and IT belonging to the enveloping algebras) which seem to be useful for a wide class of physical problems.

It is obvious that the preceding considerations concerning $L^2(SU(2))$ can be transferred to every group G being finite or compact continuous and the corresponding Hilbert space $L^2(G)$, provided one knows the unirreps, the unitary matrices U relating the unirreps to their complex conjugates and the CG coefficients for G. As already mentioned the construction of operator bases (of the type (6.7) or (8.12–14)) especially becomes of practical importance if one deals with physical systems whose state spaces are isomorphic to (invariant subspaces or coset spaces of) $L^2(G)$. Furthermore, if G is compact continuous, operator identities analogous to (8.2, 3) or (8.5) become important because they simplify the calculation of the elements of the Lie algebras in the case where one knows the elements of one of the two Lie algebras. Besides this, IT of the type (8.12, 13) can be used to construct representations of Lie algebras of 'larger' groups $G''\supset G$ (e.g. $\overline{SL}(3,R)\supset SU(2)$, where linear combinations of IT (8.12) of rank 2,

B with $B = 1, 2, 3$ can be identified with elements of the Lie algebra of $\overline{\mathrm{SL}}(3, \mathrm{R})$ of a suitably defined multiplier representation (Šijački 1975)). Apart from this, operator identities analogous to equation (6.18) should be very useful to derive relations between the Racah coefficients of G.

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